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MRC Technical Summary Report # 2245

THE SOLUTION OF THE FREE BOUNDARY PROBLEM FOR AN AXISYMMETRIC PARTIALLY PENETRATING WELL

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July 1981

(Received March 11, 1981)

Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

and

National Science Foundation
Washington, D.C. 20550

UNIVERSITY OF WISCONSIN - MADISON
MATHEMATICS RESEARCH CENTER

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ABSTRACT

The weak form of the free boundary problem for an axisymmetric partially penetrating well may be formulated as follows: find $\varphi(r) \in C^0([r_0,r_1])$ and $u \in C^0(\bar{\Omega}) \cap V^1(\Omega)$ such that

$$\int_\Omega r\nabla u \cdot \nabla v dr dz = 0 \quad \text{for all} \quad v \in K_1$$

and u satisfies appropriate boundary conditions. Here, u is related to the hydraulic head, $\varphi(r)$ is the unknown water-air interface, Ω is the region of saturated flow

$$\Omega = \{(r,z)|0 < r \le r_0, 0 < z < h\} \cup \{(r,z)|r_0 < r < r_1, 0 < z < \varphi(r)\} ,$$

K_1 is a convex set in the weighted Sobolev space $V^1(\Omega)$.

We reduce the problem to three families of variational inequalities by using a type of "Baiocchi transform", study equivalence of the three families and regularity of the solutions of the variational inequalities. Finally, we prove the existence of the solution for the well problem.

AMS (MOS) Subject Classifications: 35J20; 35J65; 35J70; 35R05; 35R35; 76S05.

Key Words: Free Boundary problem; axisymmetric well; weighted Sobolev spaces; families of variational inequalities; existence.

Work Unit Number 1 - Applied Analysis

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Sponsored by the National Science Foundation under Grant No. MCS77-26732 with support facilities provided by the U. S. Army under Contract No. DAAG29-80-C-0041.

SIGNIFICANCE AND EXPLANATION

When an axisymmetric well partially penetrates a water aquifer, the water flows through the ground towards the well. By pumping water from the well, steady flow is obtained. The flow is governed by a linear second order elliptic differential equation which degenerates at the axis of symmetry. We reformulate the problem as families of variational inequalities, and study the regularity of the solutions of these variational inequalities. Finally we prove the existence of the solution for the well problem. The variational inequality formulation suggests a new numerical method for the partially penetrating well problem.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

THE SOLUTION OF THE FREE BOUNDARY PROBLEM FOR AN AXISYMMETRIC PARTIALLY PENETRATING WELL

C. W. Cryer* and S. Z. Zhou**

Introduction

The free boundary problem for a fully penetrating well in a layer of soil of permeability $K(x,y) = \exp[f(x) + g(y)]$ has been solved by Cryer and Fetter [1979] using variational inequalities. In this paper we consider the problem for a partially penetrating well. A type of "Baiocchi Transform" (Baiocchi [1974]) is used to derive a corresponding family of variational inequalities. Existence of the solution is proved. To this end we use the theory of weighted Sobolev spaces and some results in Chang and Jiang [1978].

1. Weighted Sobolev Spaces

Our problem is governed by a degenerate elliptic equation. Degenerate elliptic equations can often be associated with a weighted Sobolev space (e.g. Murty and Stampacchia [1968], Trudinger [1973]). Various kinds of Sobolev spaces have been studied (e.g. Jakovlev [1966], Cryer [1980], Chang and Jiang [1978], Leventhal [1975] and Zhou [1980]). We recall some results.

Let A be a bounded domain in the (r,z)-plane with a locally Lipschitz boundary Γ, and with $r > 0$; $C_0^\infty(A)$ - the space of functions infinitely differentiable and with support compact in A; $C_0^\infty(A;\Gamma_1)$ - the space of functions infinitely differentiable in A and vanishing in some neighborhood

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of Γ_i, where $\Gamma_i \subset \Gamma$. $L^p(A;r)$ - the space of measurable functions satisfying

$$\|v\|_{L^p(A,r)} = \int_A r|v|^p \, drdz < \infty \quad . \tag{1.1}$$

We define weighted Sobolev spaces as follows:

$$\left.\begin{aligned} &V^0(A) = L^2(A;r) \\ &V^1(A) = \{v \mid \partial^\alpha v \in L^2(A;r),\ |\alpha| \le 1\} \\ &V^2(A) = \{v \mid \frac{1}{r}\frac{\partial v}{\partial r},\ \partial^2 v \in L^2(A;r),\ |\alpha| \le 2\} \end{aligned}\right\} \tag{1.2}$$

with norms, respectively,

$$\left.\begin{aligned} &\|v\|_{V^0(A)} = \|v\|_{L^p(A,r)} \\ &\|v\|_{V^1(A)} = \sum_{|\alpha|\le 1} \|\partial^\alpha v\|_{V^0(A)} \\ &\|v\|_{V^2(A)} = \sum_{|\alpha|\le 2} \|\partial^\alpha v\|_{V^0(A)} + \|\frac{1}{r}\frac{\partial v}{\partial r}\|_{V^0(A)} \end{aligned}\right\} . \tag{1.3}$$

Denote by $V_0^1(A)$, $V_0^1(A;\Gamma_i)$ respectively the closure of $C_0^\infty(A)$, $C_0^\infty(A;\Gamma_i)$ in $V^1(A)$.

<u>Lemma 1.1</u>. $V^0(A)$, $V^1(A)$ and $V^2(A)$ are Banach spaces.

<u>Lemma 1.2</u>. (Green's Formula). If $u, v \in V^1(A)$, then

$$\int_A ru \frac{\partial v}{\partial r} \, drdz = -\int_A v \frac{\partial (ru)}{\partial r} \, drdz + \int_\Gamma ruv \cos(n,r) ds$$

$$= -\int_A v \frac{\partial (ru)}{\partial z} \, drdz + \int_\Gamma ruv \cos(n,z) ds$$

where n is the outer normal of Γ.

<u>Lemma 1.3</u>. If A_ε is a closed subdomain of A and $\partial A_\varepsilon \cap \{r = 0\} = \emptyset$, then

$$V^1(A_\varepsilon) = H^1(A_\varepsilon) \quad .$$

Now let $\bar{A}^*$ be the three dimensional domain formed by rotating $\bar{A}$ about z-axis, and let S_i be the surface formed by rotating Γ_i about the z-axis.

Lemma 1.4. If $v(r,z) \in V^k(A)$, $k = 0,1,2$ and

$$f(x,y,z) = v(\sqrt{x^2+y^2}, z) \tag{1.4}$$

then $f \in H^k(A^*)$, where $H^k(A^*)$ is the usual Sobolev space, and A^* is the interior of $\bar{A}^*$.

Lemma 1.5. If $v \in V^1(A)$ and $\Gamma_i \quad \{r = 0\} = \emptyset$, then

$$\|f\|_{H^1(A^*)} = 2\pi \|v\|_{V^1(A)} ,$$

$$\int_{S_i} f^2 dS = 2\pi \int_{\Gamma_i} rv^2 ds \quad .$$

By using Lemma 1.5 and results in Sobolev [§10, 1950] we obtain:

Lemma 1.6. If $v \in V_0^1(A;\Gamma_i)$ and

$$\text{mes}[\Gamma_i \setminus (\Gamma_i \cap \{r = 0\})] > 0$$

then

$$\|v\|^2_{V^1(A)} \leq C \int_A [(\frac{\partial v}{\partial r})^2 + (\frac{\partial v}{\partial z})^2] r \, drdz$$

where C does not depend on v.

2. Descritption of the Problem

The problem to be considered is shown in Figure 2.1.

A cylindrical well of radius r_0 partially penetrates a layer of soil of depth H and radius r_1. Take the axis of symmetry as the z-axis. The bottom of the soil layer is impermeable. The distance of the well bottom from the bottom of the soil layer is h. We assume that the soil layer is homogeneous and isotropic; that the water is incompressible; that the flow is irrotational and steady (in particular the height of water on the outer boundary of the soil and in the well is respectively H and h_w); that the permeability $k(r,z) \equiv 1$.

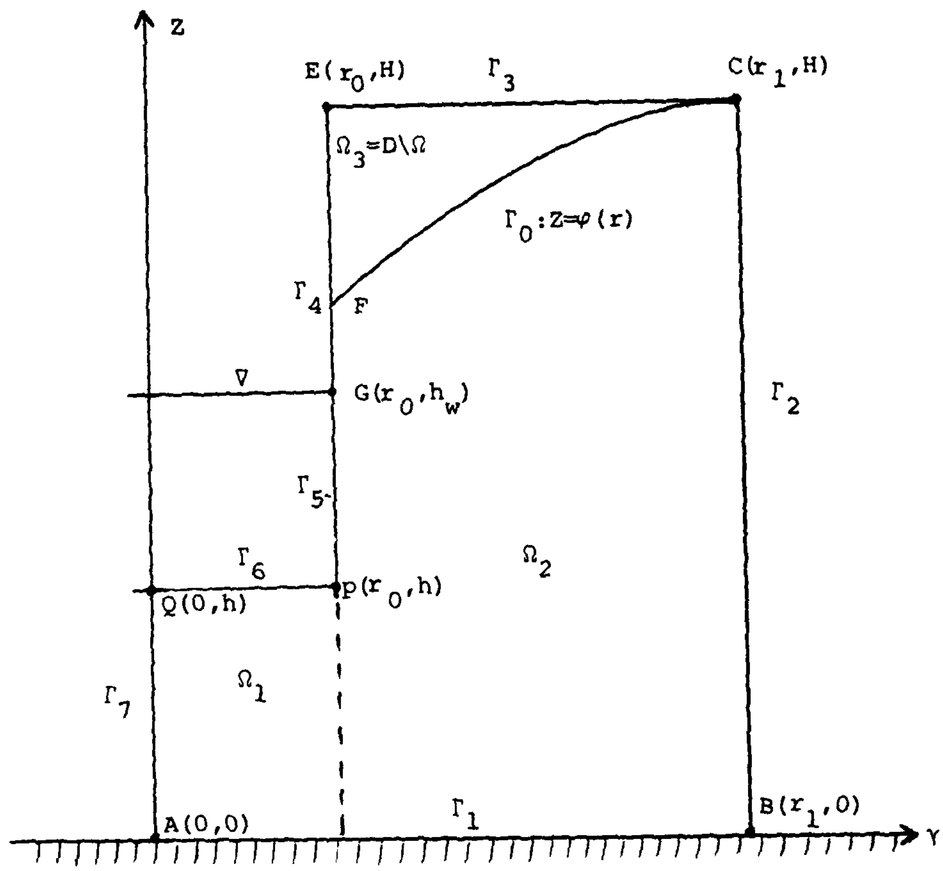


Figure 2.1

The cross section of the soil layer is

$$D = \Omega_1 \cup \Omega_2 \cup \Omega_3 \tag{2.1}$$

where

$$\Omega_1 = \{(r,z) | 0 < r < r_0,\ 0 < z < h\}$$

$$\Omega_2 = \{(r,z) | r_0 < r < r_1,\ 0 < z < \varphi(r)\}$$

$$\Omega_3 = \{(r,z) | r_0 < r < r_1,\ \varphi(r) < z < H\}$$

and $z = \varphi(r)$ is the boundary between the wet region $\Omega = \Omega_1 \cup \Omega_2$ and dry region Ω_3. It is called the free boundary as it is the unknown part of $\partial\Omega$.

Denote by $p(r,z)$ and $u(r,z)$ respectively the pressure at point (r,z) of D (the atmospheric pressure being measured by zero) and the hydraulic head, then we have

$$u(r,z) = p(r,z) + z \quad \text{in} \quad \Omega \ . \tag{2.2}$$

It follows from Darcy's law and the equation of continuity that (see Hantush [1964], or Cryer [1976, p. 86])

$$Lu = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0 \quad \text{in} \quad \Omega \ . \tag{2.3}$$

We introduce the notation

$$\begin{aligned}
\Gamma_1 &= \{(r,z) \mid 0 < r < r_1,\ z = 0\} \\
\Gamma_2 &= \{(r,z) \mid r = r_1,\ 0 < z < H\} \\
\Gamma_3 &= \{(r,z) \mid r_0 < r < r_1,\ z = H\} \\
\Gamma_4 &= \{(r,z) \mid r = r_0,\ h_w < z < H\} \\
\Gamma_5 &= \{(r,z) \mid r = r_0,\ h < z < h_w\} \\
\Gamma_6 &= \{(r,z) \mid 0 < r < r_0,\ z = h\} \\
\Gamma_7 &= \{(r,z) \mid r = 0,\ 0 < z < h\} \\
\Gamma_0 &= \{(r,z) \mid r_0 < r < r_1,\ z = \varphi(r)\}.
\end{aligned}$$

Then $u(r,z)$ satisfies the following boundary conditions:

$$\left.\begin{array}{lll}
u = H & \text{on } \Gamma_2 & \text{(constant hydraulic head)} \\
u = z & \text{on } \Gamma_0 \cup (\Gamma_4 \cap \partial\Omega) & \text{(interface with air)} \\
u = h_w & \text{on } \Gamma_5 \cup \Gamma_6 & \text{(interface with water at rest)}
\end{array}\right\} \tag{2.4}$$

$$\left.\begin{array}{llll} \frac{\partial u}{\partial n} = 0 & \text{on} & \Gamma_0 \cup \Gamma_1 & \text{(streamline)} \\ \frac{\partial u}{\partial n} = 0 & \text{on} & \Gamma_7 & \text{(symmetry)} \end{array}\right\} \quad . \tag{2.5}$$

Now we can state our problem in weak form.

<u>Problem (PPW)</u>

Given the domain as in (2.1), and a real number h_w such that $h < h_w < H$, find functions $\varphi(r)$ and $u(r,z)$ such that u satisfies (2.4) and

$$\varphi \in C^0([r_0,r_1]),\ \varphi(r_1) = H,\ \varphi(r_0) > h_w \quad , \tag{2.6}$$

$$\varphi \text{ is strictly increasing} \quad , \tag{2.7}$$

$$u \in V^1(\Omega) \cap C^0(\overline{\Omega}) \quad , \tag{2.8}$$

$$\int_\Omega r\nabla u \cdot \nabla v \; drdz = 0 \quad \text{for all} \quad v \in K_1 \tag{2.9}$$

where

$$\Omega = \Omega_1 \cup \{(r,z) \in D | r > r_0,\ 0 < z < \varphi(r)\}$$

$$K_1 = \{v \in V^1(\Omega) | v = 0 \quad \text{on} \quad \Gamma_2 \cup (\Gamma_4 \cap \partial\Omega) \cup \Gamma_5 \quad \Gamma_6\} \quad .$$

<u>Remark 2.1.</u> This problem can be regarded as a plane problem with permeability $K = \exp(\ell n\ r)$, but it is not covered by the work of Benci [1974] because $\ell n\ r \notin H^{1,2+\mu}([0,r_1])$. Also, it can not be included in Alt [1979] as a two-dimensional problem. Rama and Das [1976] solved this problem by numerical methods.

<u>Remark 2.2.</u> Chang and Jiang [1978] have solved a similar problem by using so-called "Sequence of set-valued mappings" instead of the method of variational inequalities. Further results about obstacle problems have been obtained recently by Chang [1980]. We use some results on linear equations given by Chang and Jiang [1978]. But we solve our problem by using the method of variational inequalities because the corresponding numerical method is more

convenient, and because in our case the boundary conditions and right term of the nonlinear equation for the Baiocchi function w are different from those in Chang and Jiang [1978].

3. The Baiocchi Function w and its Properties

Assuming a priori the existence of the solution u of (PPW), set (see Baiocchi [1974], [1976], [1978])

$$\bar{u}(r,z) = \begin{cases} u(r,z) & \text{in } \bar{\Omega} \\ z & \text{in } \bar{D}\backslash\bar{\Omega} \end{cases} \tag{3.1}$$

$$w(r,z) = \int_0^z [\bar{u}(r,t) - t]\,dt \quad . \tag{3.2}$$

Remark 3.1. We can not use the transform

$$\tilde{w}(r,z) = \int_z^{\varphi_1(r)} [\bar{u}(r,t) - t]\,dt \quad ,$$

where $\varphi_1(r) = H$ for $r_0 < r < r_1$ and $\varphi_1(r) = h$ for $0 < r \leq r_0$, since $\frac{\partial\tilde{w}}{\partial r} \notin C^1(\bar{D})$. This is obvious physically. (Cf. Lemma 3.6).

Now we derive some properties of w and u.

Lemma 3.1.
$$Lu = 0 \quad \text{in} \quad \Omega \tag{3.3}$$

$$u \text{ is analytic in } \bar{\Omega}\backslash\{\Gamma_0, A, B, C, F, G, P, Q\} \tag{3.4}$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \Gamma_1 \cup \Gamma_7 \tag{3.5}$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{in the weak sense on} \quad \Gamma_0 \quad . \tag{3.6}$$

Proof. On writing (2.9) for any $v \in C_0^\infty(\Omega)$, we obtain (3.3) in the sense of distributions. From classical results on the regularity of the variational solutions of elliptic equations in the interior and on the smooth parts of the boundary (see for instance Lions and Magenes [1972, §9, Ch. 2]) it follows that

$$u \text{ is analytic in } \bar{\Omega}\setminus\{\Gamma_0, \Gamma_7, A, B, C, F, G, P, Q\} .$$

Denote by $\bar{\Omega}^*$ the three-dimensional axisymmetric domain with cross-section $\bar{\Omega}$. Then $u^*(x,y,z) = u(\sqrt{x^2 + y^2}, z)$ is a solution of the equation $\Delta u = 0$ in Ω^* where Ω^* is the interior of $\bar{\Omega}^*$. Hence u^* and u are analytic on Γ_7 because Γ_7 is in Ω^*, and (3.4) is valid. (2.9) implies (3.5) and (3.6) in the weak sense. (3.3), (3.5) are satisfied also in the classical sense thanks to (3.4).

Q.E.D.

Remark 3.2. It follows from the three-dimensional argument above and maximum principle that if $Lv \geq 0$ in Ω with $v \in C^0(\bar{\Omega})$ then

$$v|_{\Gamma_7} < \max_{\bar{\Omega}} v$$

and if in addition $\frac{\partial v}{\partial z} \leq 0$ at the point Q then

$$v|_Q < \max_{\bar{\Omega}} v \quad .$$

Similar results are valid for min v if $Lv \geq 0$.

Lemma 3.2. $\quad u(r,z) > z \text{ in } \Omega .\qquad (3.7)$

Proof. Set $v = u - z$. Then we have

$$\begin{cases} Lv = 0 & \text{in } \Omega \\ v = 0 & \text{on } \Gamma_0 \quad (\partial\Omega \cap \Gamma_4) \\ v = H - z & \text{on } \Gamma_2 \\ v = h_w - z & \text{on } \Gamma_5 \cup \Gamma_6 \\ \frac{\partial v}{\partial n} = +1 & \text{on } \Gamma_1 \\ \frac{\partial v}{\partial n} = 0 & \text{on } \Gamma_7 \quad . \end{cases}$$

Since L is elliptic and $v \in C^0(\bar{\Omega})$, v attains its minimum, m say, in $\bar{\Omega}$ at a point $p^* \in \partial\Omega$. But $p^* \notin \Gamma_1$ (by Hopf principle) and $p^* \notin \Gamma_7$ (by Remark 3.2). So $p^* \in \partial\Omega \setminus (\Gamma_1 \cup \Gamma_7)$. Hence m is zero. It follows from the strong maximum principle that $v > 0$ (i.e. $u > z$) in Ω.

Q.E.D.

<u>Lemma 3.3.</u>

$$\bar{u} \in V^1(D) \cap C^0(\bar{D}) \tag{3.8}$$

$$L\bar{u} = -\frac{\partial \Phi_\Omega}{\partial z} \text{ in the sense of distributions} \tag{3.9}$$

where Φ_Ω is the characteristic function of Ω in D.

<u>Proof</u>. By (2.4), (2.8) and (3.1) it is easy to see that $\bar{u} \in C^0(\bar{D})$. For any $\psi \in C_0^\infty(D)$ we have

$$\int_D \bar{u} \frac{\partial \psi}{\partial z} drdz = \int_{\Omega_1} + \int_{\Omega_2} + \int_{\Omega_3}$$

$$= \int_0^{r_0} dr \int_0^h u \frac{\partial \psi}{\partial z} dz + \int_{r_0}^{r_1} dr \int_0^{\varphi(r)} u \frac{\partial \psi}{\partial z} dz + \int_{r_0}^{r_1} dr \int_{\varphi(r)}^H a \frac{\partial \psi}{\partial z} dz$$

$$= -\int_{\Omega_1} \psi \frac{\partial u}{\partial z} drdz - \int_{\Omega_2} \psi \frac{\partial u}{\partial z} drdz - \int_{\Omega_3} \psi \, drdz \quad \text{(Integration by parts)}$$

$$= \int_D \psi v \, drdz$$

where

$$v = \begin{cases} \dfrac{\partial u}{\partial z} & \text{in } \Omega \\ 1 & \text{in } D \setminus \Omega \end{cases}.$$

Hence

$$\frac{\partial \bar{u}}{\partial z} = \begin{cases} \dfrac{\partial u}{\partial z} & \text{in } \Omega \\ 1 & \text{in } D \setminus \Omega \end{cases}.$$

Similarly we have

$$\frac{\partial \bar{u}}{\partial r} = \begin{cases} \frac{\partial u}{\partial r} & \text{in } \Omega \\ 0 & \text{in } D\backslash\Omega \end{cases} .$$

Clearly, $\bar{u} \in V^1(D)$.

Now we prove (3.9). If $\psi \in C_0^\infty(D)$ then $\psi \in K_1$. Hence we have for the distribution $L\bar{u}$ and every $\psi \in C_0^\infty(D)$

$$\langle L\bar{u}, r\psi\rangle = -\int_D r\nabla\bar{u}\cdot\nabla\psi \, drdz$$

$$= -\int_\Omega r\nabla u\cdot\nabla\psi \, drdz - \int_{D\backslash\Omega} \frac{\partial\psi}{\partial z} r \, drdz$$

$$= -\int_D \frac{\partial\psi}{\partial z}(1 - \Phi_\Omega)r \, drdz = \langle\frac{\partial}{\partial z}(1 - \Phi_\Omega), r\psi\rangle$$

$$= -\langle\frac{\partial\Phi_\Omega}{\partial z}, r\psi\rangle .$$

It is just (3.9).

Q.E.D.

Proposition 3.4. Let w be defined by (3.2). Then

$$Lw = -\Phi_\Omega \text{ in the sense of distributions.} \tag{3.10}$$

Proof. Since $\bar{u} \in C^0(\bar{D})$, we have

$$\frac{\partial w}{\partial z} = \bar{u}(r,z) - z . \tag{3.11}$$

So in the distribution sense we obtain (by (3.9))

$$\frac{\partial}{\partial z}(Lw) = L\left(\frac{\partial w}{\partial z}\right) = L\bar{u} - Lz = -\frac{\partial\Phi_\Omega}{\partial z} .$$

Hence (Schwartz [1973, §5, Ch. 2])

$$Lw + \Phi_\Omega = T(r) \otimes 1(z) .$$

Since u is analytic in $\Omega \cup \Gamma_1$ and $\frac{\partial u}{\partial z} = 0$ on Γ_1, we have in Ω:

$$Lw = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2}$$

$$= \int_0^z \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right)dt + \frac{\partial u(r,z)}{\partial z} - 1$$

$$= \int_0^z Lu \, dt + \frac{\partial u(r,0)}{\partial z} - 1 = 0 .$$

Accordingly, $T(r) \equiv 0$ and (3.10) is valid.

Q.E.D.

Proposition 3.5. w is a solution of the equation with discontinuous nonlinearities:

$$Lw = \begin{cases} 0 & \text{for } \frac{\partial w}{\partial z} = 0 \\ -1 & \text{for } \frac{\partial w}{\partial z} > 0 \end{cases} . \tag{3.12}$$

Proof. It is enough to prove that

$$\frac{\partial w}{\partial z} > 0 \quad , \quad \text{if} \quad (r,z) \in \Omega \quad ,$$

$$\frac{\partial w}{\partial z} = 0 \quad , \quad \text{if} \quad (r,z) \in D \setminus \Omega \quad .$$

But this is obvious thanks to Lemma 3.2 and (3.11).

Q.E.D.

Lemma 3.6.

$$\frac{\partial w}{\partial r} = \int_0^z \frac{\partial \bar{u}}{\partial r}\, dt \tag{3.13}$$

$$r\,\frac{\partial w(r,H)}{\partial r} = \text{constant} = 0 \quad \text{for} \quad r \in\,]r_0,r_1[\quad . \tag{3.14}$$

Proof. For any $\psi \in C_0^\infty(D)$ we have

$$\int_{\Omega_1} w\,\frac{\partial \psi}{\partial r}\, drdz = \int_0^h dz \int_0^{r_0} \left(\int_0^z [\bar{u}(r,t) - t]\,\frac{\partial \psi}{\partial r}\, dr\right.$$

$$= \int_0^h dz \int_0^z dt \int_0^{r_0} [\bar{u}(r,t) - t]\,\frac{\partial \psi}{\partial r}\, dr$$

$$= \int_0^h dz \int_0^z [\bar{u}(r,t) - t]\,\psi|_{r=r_0} dt - \int_{\Omega_1} \psi\left(\int_0^z \frac{\partial \bar{u}}{\partial r}\, dt\right) drdz$$

(integration by parts).

Similarly, we have

$$\int_{D\setminus\Omega_1} w \frac{\partial\psi}{\partial r}\,drdz = -\int_0^H dz \int_0^z [\bar{u}(r,t) - t]\psi|_{r=r_0} dt - \int_{D\setminus\Omega_1} \psi(\int_0^z \frac{\partial\bar{u}}{\partial r} dt)drdz \quad .$$

Since $\psi = 0$ on $\{(r,z)|r=r_0, h < z < H\}$, we obtain immediately

$$\int_D w \frac{\partial\psi}{\partial r}\,drdz = -\int_D \psi(\int_0^z \frac{\partial\bar{u}(r,t)}{\partial r} dt)drdz \quad \text{for any} \quad \psi \in C_0^\infty(D)$$

(3.13) has been proved. Now set $f(r) = r\,\frac{\partial w(r,H)}{\partial r}$. Then for any $\psi(r) \in C_0^\infty(]r_0,r_1[)$ we have

$$\begin{aligned}\int_{r_0}^{r_1} \frac{\partial\psi}{\partial r}(r)f(r)dr &= \int_{r_0}^{r_1} \frac{\partial\psi}{\partial r}\, r\, \frac{\partial w(r,H)}{\partial r}\, dr \\ &= \int_{r_0}^{r_1} \frac{\partial\psi}{\partial r}\, r(\int_0^H \frac{\partial\bar{u}(r,t)}{\partial r} dt)dr \\ &= \int_{r_0}^{r_1} dr \int_0^H \frac{\partial\psi}{\partial r}\, r\, \frac{\partial\bar{u}(r,z)}{\partial r}\, dz \\ &= \int_\Omega r\nabla u\cdot\nabla\psi_1\, drdz\end{aligned}$$

where

$$\psi_1(r,z) = \begin{cases} \psi(r) & \text{for} \quad r \in]r_0,r_1[\\ 0 & \text{for} \quad r \in]0,\ r_0] \end{cases} \quad .$$

Clearly $\psi_1 \in K_1$. Hence it follows from (2.9) that

$$\int_{r_0}^{r_1} \frac{\partial\psi}{\partial r} f\, dr = 0 \qquad \forall\, \psi \in C_0^\infty(]r_0,r_1[)$$

so $f(r)$ is a constant for $r \in]r_0,r_1[$, which we denote by q.

Q.E.D.

Remark 3.3. Physically, $2\pi q$ is the discharge of the well.

Proposition 3.7. Let

$$g_q = \begin{cases} 0 & \text{on } \Gamma_1 \\ Hz - \dfrac{z^2}{2} & \text{on } \Gamma_2 \\ \dfrac{H^2}{2} + q \,\ell n \dfrac{r}{r_1} & \text{on } \Gamma_3 \cup \Gamma_4 \\ \dfrac{H^2}{2} + q \,\ell n \dfrac{r_0}{r_1} - \dfrac{(h_w - z)^2}{2} & \text{on } \Gamma_5 \end{cases} \tag{3.15}$$

$$g_N = \begin{cases} h_w - h & \text{on } \Gamma_6 \\ 0 & \text{on } \Gamma_7 \end{cases} \quad . \tag{3.16}$$

Then
$$w(r,z) = g_q \quad \text{on } \Gamma_D$$

$$\frac{\partial w}{\partial n} = g_N \quad \text{on } \Gamma_N$$

where $\Gamma_D = \bigcup_{i=1}^{5} \Gamma_i$, $\Gamma_N = \Gamma_6 \cup \Gamma_7$.

Proof. Thanks to (3.2) and (2.4) we have clearly

$$w = 0 \quad \text{on } \Gamma_1, \; w = Hz - \frac{z^2}{2} \quad \text{on } \Gamma_2 \quad .$$

Hence $w(r_1,H) = H^2/2$. Solving the ordinary differential equation $r \dfrac{\partial w(r,H)}{\partial r} = q$ we obtain

$$w = \frac{H^2}{2} + q \,\ell n \frac{r}{r_1} \quad \text{on } \Gamma_3 \quad .$$

On Γ_4 we have $\dfrac{\partial w}{\partial z} = \bar{u} - z = 0$. Hence

$$w(r_0,z) = w(r_0,H) = \frac{H^2}{2} + q \,\ell n \frac{r_0}{r_1} \quad .$$

On Γ_5 we have $\dfrac{\partial w}{\partial z} = h_w - z$ and

$$w = w(r_0,h_w) + \int_{h_w}^{z} \frac{\partial w}{\partial z}\, dz = \frac{H^2}{2} + q \,\ell n \frac{r_0}{r_1} - \frac{(h_w - z)^2}{2}$$

(3.16) is obvious.

Proposition 3.8.

$$w(r,z) = g_q(r,H) \quad \text{in} \quad D\backslash\Omega \tag{3.17}$$

$$w(r,z) < g_q(r,H) \quad \text{in} \quad \Omega\backslash\Omega_1 \quad . \tag{3.18}$$

Proof. (3.17) follows from (3.1), (3.2) and Proposition 3.7. (3.18) follows from (3.17), Lemma 3.2 and the fact that $u \in C^0(\bar{\Omega})$.

Q.E.D.

Remark 3.4. We obtain another form of the nonlinear equation for w:

$$Lw = \begin{cases} 0 & \text{in} \quad \{w = g_q(r,H)\} \\ -1 & \text{in} \quad \Omega_1 \cup \{w < g_q(r,H)\} \end{cases} .$$

Proposition 3.9.

$$w \in V^2(D) \quad . \tag{3.20}$$

Proof. By (3.2) and (2.8) we have

$$\frac{\partial w}{\partial z}, \frac{\partial^2 w}{\partial z^2}, \frac{\partial^2 w}{\partial r \partial z} \in V^0(D) \quad . \tag{3.21}$$

Differentiating (3.13) we obtain

$$\frac{\partial^2 w}{\partial z \partial r} = \frac{\partial \bar{u}}{\partial r} \in V^0(D) \quad . \tag{3.22}$$

Now we prove that

$$\frac{\partial w}{\partial r} \in V^0(D) \quad . \tag{3.23}$$

In fact, we have

$$\int_D r\left(\int_0^z \frac{\partial \bar{u}}{\partial r}\, dt\right)^2 drdz \le \int_D rH\left[\int_0^z \left(\frac{\partial \bar{u}}{\partial r}\right)^2 dt\right] drdz \quad \text{(Schwartz inequality)}$$

$$= H\left[\int_0^{r_0} dr \int_0^h dz \int_0^z r\left(\frac{\partial \bar{u}}{\partial r}\right)^2 dt + \int_{r_0}^{r_1} dr \int_0^H dz \int_0^z r\left(\frac{\partial \bar{u}}{\partial r}\right)^2 dt\right]$$

$$\le H\left[h \int_0^{r_0} dr \int_0^h r\left(\frac{\partial \bar{u}}{\partial r}\right)^2 dt + H \int_{r_0}^{r_1} dr \int_0^H r\left(\frac{\partial \bar{u}}{\partial r}\right)^2 dt\right]$$

$$\le H^2 \int_D r\left(\frac{\partial \bar{u}}{\partial r}\right)^2 drdz < \infty \quad .$$

At last we prove that

$$\frac{1}{r}\frac{\partial w}{\partial r}, \frac{\partial^2 w}{\partial r^2} \in V^0(D) \quad . \tag{3.24}$$

By (3.10) we have, as distributions,

$$\frac{\partial^2 w}{\partial r^2} = -\Phi_\Omega - \frac{\partial^2 w}{\partial z^2} - \frac{1}{r}\frac{\partial w}{\partial r} \quad .$$

Hence it is sufficient to prove that

$$\frac{1}{r}\frac{\partial w}{\partial r} \in V^0(D) \quad . \tag{3.25}$$

Let $R^* = \{(r,z) | 0 < r < \frac{r_0}{2}, 0 < z < h\}$, $Q' = (\frac{r_0}{2},h)$, $A' = (\frac{r_0}{2},0)$, $f(z) = u(\frac{r_0}{2},z)$. Then $u|_{\bar{R}^*} \in C^\infty(\bar{R}^* \setminus \{A,Q\}) \cap C^0(\bar{R}^*) \cap V^0(R^*)$ (by Lemma 3.1), and $u|_{R^*}$ is the solution of the boundary value problem

$$\begin{cases} Lu = 0 & \text{in } R^* \\ u|_{z=h} = h_w, \ \frac{\partial u}{\partial z}|_{z=0} = 0 & \\ u|_{r=\frac{r_0}{2}} = f(z), \ u \ \underline{\text{is bounded near}} \ r = 0 & . \end{cases}$$

By using the method of separating variables we obtain that

$$I_0(x) = \sum_{k=0}^{\infty} \frac{1}{k!}\frac{1}{\Gamma(k+1)}\left(\frac{x}{2}\right)^{2k} \quad . \tag{3.26}$$

It is easy to show that $\frac{\partial u}{\partial r} = 0(r)$ as $r \to 0$. Hence $\frac{1}{r}\frac{\partial w}{\partial r} = 0(1)$ as $r \to 0$, and $\frac{1}{r}\frac{\partial w}{\partial r}|_{R^*} \in V^0(R^*)$. Now (3.25) is clear.

Q.E.D.

Proposition 3.10. $$w \in C^1(\bar{D}) \quad . \tag{3.27}$$

In order to prove this proposition we cite a theorem in Chang and Jiang [1978], the proof of which is given in the Appendix.

Theorem 3.A. If $v \in V^2(D)$ and $Lv = f \in L^p(D;r)$ $(p > 6)$, $v|_{\Gamma_D} = \frac{\partial v}{\partial n}|_{\Gamma_N} = 0$, then $v \in C^\beta(\bar{D})$ $(\beta < \frac{3}{2})$.

Proof of Proposition 3.10:

At first we construct a function v_q such that

$$\left.\begin{aligned} & v_q|_{\Gamma_D} = g_q \\ & \frac{\partial v_q}{\partial n}|_{\Gamma_N} = g_N \\ & v_q \in V^2(D) \cap C^1(\bar{D}), \; Lv_q \in L^\infty(D;r) \end{aligned}\right\} . \tag{3.28}$$

To this end we set (cf. Baiocchi et al. [1973], Chang and Jiang [1978])

$$v_q = v_1 + qv_2 \tag{3.29}$$

where

$$v_1 = \begin{cases} f_0(z) & 0 < r < \frac{r_0+r_1}{2} \\ [f_1(z) - f_0(z)]\left(\frac{2r-r_1-r_0}{r_1-r_0}\right)^2 + f_0(z) & \frac{r_0+r_1}{2} < r < r_1 \end{cases} \tag{3.30}$$

$$v_2 = \begin{cases} f_2(r)\,[1 - (\frac{z-h}{h})^2] & 0 < z < h \\ f_2(r) & h < z < H \end{cases} \tag{3.31}$$

and

$$f(z) = \begin{cases} \frac{H^2}{2} & h_w < z < H \\ \frac{H^2}{2} - \frac{(h_w-z)^2}{2} & 0 < z < h_w \end{cases}$$

$$f_0(z) = \begin{cases} f(z) & h < z < H \\ f(z)[1 - (\frac{z-h}{h})^2] & 0 < z < h \end{cases}$$

$$f_1(z) = Hz - \frac{z^2}{2}$$

$$f_2(r) = \begin{cases} \ln \frac{r_0}{r_1} + 3(\frac{r}{r_0})^2 - 2(\frac{r}{r_0})^3 - 1 & 0 < r < r_0 \\ \ln \frac{r}{r_1} & r_0 < r < r_1 \end{cases} \quad . \tag{3.32}$$

It is readily verified that

$$v_1|_{\Gamma_D} = \begin{cases} 0 & \text{on } \Gamma_1 \\ Hz - \frac{z^2}{2} & \text{on } \Gamma_2 \\ \frac{H^2}{2} & \text{on } \Gamma_3 \cup \Gamma_4 \\ \frac{H^2}{2} - \frac{(h_w - z)^2}{2} & \text{on } \Gamma_5 \end{cases} \tag{3.33}$$

$$\frac{\partial v_1}{\partial n}|_{\Gamma_N} = g_N \tag{3.34}$$

$$v_2|_{\Gamma_D} = \begin{cases} 0 & \text{on } \Gamma_1 \cup \Gamma_2 \\ \ln \frac{r}{r_1} & \text{on } \Gamma_3 \cup \Gamma_4 \cup \Gamma_5 \end{cases} \tag{3.35}$$

$$\frac{\partial v_2}{\partial n}|_{\Gamma_N} = 0 \tag{3.36}$$

$$v_1,\ v_2 \in V^2(D) \cap C^1(\bar{D}) \tag{3.37}$$

$$Lv_1,\ Lv_2 \in L^\infty(D;r) \quad . \tag{3.38}$$

Hence (3.28) is valid.

Now we set $v = w - v_q$, then $v \in V^2(D)$ (by (3.20) and (3.28)), and

$$Lv = -\Phi_\Omega - Lv_q \in L^\infty(D;r)$$

$$v|_{\Gamma_D} = \frac{\partial v}{\partial n}|_{\Gamma_N} = 0$$

(3.27) follows from Theorem 3.A immediately.

Q.E.D.

<u>Proposition 3.11</u>

$$q = \lim_{r \to r_0+0} \frac{r_0}{r-r_0} \int_h^H [\bar{u}(r,t) - \bar{u}(r_0,t)]dt$$
$$+ \lim_{z \to h-0} \frac{1}{z-h} \int_0^{r_0} [\bar{u}(r,z) - \bar{u}(r,h)]r\,dr \quad . \tag{3.39}$$

<u>Proof:</u> By the mean value theorem of differentiation we have that for any $r \in]r_0,r_1[$ there exists $\xi \in]r_0,r[$ such that

$$\int_h^H [\bar{u}(r,t) - \bar{u}(r_0,t)]dt = \int_h^H [\bar{u}(r,t) - t]dt - \int_h^H [\bar{u}(r_0,t) - t]dt$$

$$= w(r,H) - w(r,h) - [w(r_0,H) - w(r_0,h)]$$

$$= [\frac{\partial w(\xi,H)}{\partial r} - \frac{\partial w(\xi,h)}{\partial r}]\,(r - r_0) \quad .$$

It follows from (3.27) and (3.15) that

$$\lim_{r \to r_0+0} \frac{r_0}{r-r_0} \int_h^H [\bar{u}(r,t) - \bar{u}(r_0,t)]dt = q - r_0 \frac{\partial w(r_0,h)}{\partial r} \quad . \tag{3.40}$$

On the other hand, for any $z \in]0,h[$ there exists a $\eta \in]z,h[$ such that

$$\frac{1}{z-h} \int_0^{r_0} [\bar{u}(r,z) - \bar{u}(r,h)]r\,dr = \int_0^{r_0} \frac{\partial \bar{u}(r,\eta)}{\partial z} r\,dr \quad . \tag{3.41}$$

Let $D_1 = \{(r,z) | 0 < r < r_0, 0 < z < \eta\}$. It is easy to show by Green's formula that

$$0 = \int_{D_1} L\bar{u} \cdot r \, drdz = \int_0^{\eta} r_0 \frac{\partial \bar{u}(r_0,z)}{\partial r} dz - \int_0^{r_0} r \frac{\partial \bar{u}(r,\eta)}{\partial z} dr \ .$$

So we have

$$\int_0^{r_0} \frac{\partial \bar{u}(r,\eta)}{\partial z} r \, dr = r_0 \int_0^{\eta} \frac{\partial \bar{u}(r_0,z)}{\partial r} dz - r_0 \frac{\partial w(r_0,\eta)}{\partial r} \ . \tag{3.42}$$

Now we obtain (3.39) by (3.40) - (3.42) and (3.27).

Q.E.D.

Remark 3.5. Physically, (3.39) means that the total discharge to the well consists of two parts: one is across the wall of the well, another is across the bottom.

4. Variational Inequalities (VI) Satisfied by w; Regularity of the Solution of VI

Define functions for every $v \in V^1(D)$:

$$v' = \begin{cases} v & \text{in } \Omega_1 \quad \{v \leq g_{qH}\} \\ g_q(r,H) & \text{in } \{v > g_{qH}\} \end{cases} \tag{4.1}$$

$$v'' = \begin{cases} 0 & \text{in } \Omega_1 \quad \{v \leq g_{qH}\} \\ v - g_q(r,H) & \text{in } \{v > g_{qH}\} \end{cases} \tag{4.2}$$

where $\{v \leq g_{qH}\} = \{(r,z) \in D | r > r_0, \ v(r,z) \leq g_q(r,H)\}$

$\{v > g_{qH}\} = \{(r,z) \in D | r > r_0, \ v(r,z) > g_q(r,H)\}$.

Then, clearly, we have

$$v = v' + v'',\ v'' \geq 0,\ v' \leq g_q(r,H) \quad \text{for} \quad r \in]r_0, r_1[,\ v',\ v'' \in V^0(D) \quad . \tag{4.3}$$

Let

$$K_q = \{v \in V^1(D) \mid v = g_q \quad \text{on} \quad \Gamma_D\} \quad . \tag{4.4}$$

We have

Theorem 4.1. If u is a solution of (PPW), then w defined by (3.2) is a solution of the VI:

$$\begin{cases} w \in K_q \\ \int_D r\nabla w \cdot \nabla(v-w)drdz - (h_w-h) \int_0^{r_0} r(v-w)|_{z=h}dr - \int_D (v'-w')r\ dr\ dz \geq 0 \\ \qquad\qquad\qquad\qquad \text{for} \quad v \in K_q \quad . \end{cases} \tag{4.5}$$

Proof: By (3.20) we have $w \in V^2(D)$. Apply to w and to any $v \in K_q$ the following Green's formula:

$$\begin{aligned} &\int_D r\nabla w \cdot \nabla(v-w)drdz \\ &\quad = -\int_D rLw \cdot (v-w)drdz + \int_{\Gamma_D} r(v-w)\frac{\partial w}{\partial n}ds + \int_{\Gamma_N} r(v-w)\frac{\partial w}{\partial n}ds \\ &\quad = \int_D r\Phi_\Omega \cdot (v-w)drdz + (h_w-h)\int_0^{r_0}(v-w)|_{z=h}r\ dr \\ &\quad = \int_\Omega r(v-w)drdz + (h_w-h)\int_0^{r_0}(v-w)|_{z=h}r\ dr \\ &\quad \geq \int_\Omega r(v'-w')drdz + (h_w-h)\int_0^{r_0}(v-w)|_{z=h}r\ dr \quad \begin{pmatrix} v'' \geq 0 & \text{and} \\ w'' = 0 & \end{pmatrix} \quad . \end{aligned}$$

But $-\int_{D\ \Omega}(v'-w')r\ drdz = -\int_{D\setminus\Omega}[v'-g_q(r,H)]r\ drdz \geq 0$ (by (3.17)). Hence (4.5) is valid.

Q.E.D.

Remark 4.1. It is easily seen (by (3.17) and (3.18)) that w is also a solution of the VI

$$w \in K_q^*$$

$$\int_D r\nabla w \cdot \nabla(v-w)\,drdz - (h_w-h)\int_0^{r_0}(v-w)|_{z=h}r\,dr - \int_D r(v-w)drdz \geq 0 \quad (4.6)$$

$$\text{for } v \in K_q^*$$

where

$$K_q^* = \{v \in V^1(D)\,|\,v = g_q \text{ on } \Gamma_D,\ v \leq g_q(r,H) \text{ in } D\backslash\Omega_1\} . \quad (4.7)$$

Remark 4.2. Noting (3.7) we have that w is also a solution of VI

$$w \in K_q^{**}$$

$$\int_D r\nabla w \cdot \nabla(v-w)\,drdz - (h_w-h)\int_0^{r_0}(v-w)|_{z=h}r\,dr - \int_D (v-w)r\,drdz \geq 0 \quad (4.8)$$

$$\text{for } v \in K_q^{**}$$

where

$$K_q^{**} = \{v \in K_q^*\,|\,v \geq 0 \text{ in } D\} . \quad (4.9)$$

Remark 4.3. For numerical solutions (4.8) is the most convenient VI. By the well-known result (Lions [1971, p. 9]), Problem (4.8) reduces a minimization problem on a convex set as follows: find $w \in K_q^{**}$ such that

$$J(w) = \min_{v \in K_q^{**}} J(v) \quad (4.8')$$

where

$$J(v) = \int_D r|\nabla v|^2 drdz - (h_w-h)\int_0^{r_0} v_{z=h}r\,dr - 2\int_D vr\,drdz .$$

(4.8') is the basis of numerical solutins to (PPW) by using VI's.

For $q \in R$, (4.4) is a family of VI's. So are (4.6) and (4.8). Now we study these families.

Proposition 4.2. $\forall q \in R$, (4.4) has unique solution w_q.

Proof: Set $V = \{v \in V^1(D) | v = 0 \text{ on } \Gamma_1\} \equiv V_0^1(D;\Gamma_1)$

$$a(u,v) = \int_D r\nabla u \cdot \nabla v \, drdz$$

$$f(v) = \int_D v' r \, drdz + (h_w - h) \int_0^{r_0} rv|_{z=h} dr \quad .$$

Then V is a Hilbert space with inner product $(u,v)_V = \int_D r(uv + \nabla u \cdot \nabla v) dr\, dz$; K_q is a closed, convex, non-empty (e.g. $v_q \in K_q$; see (3.28)) subset of V; $a(u,v)$ is a bilinear, continuous and coercive form on $V \times V$ (by Lemma 1.6); and it is easy to show that $f(v)$ is a convex, continuus functional on V with $f(v) \not\equiv -\infty$ and $f(v) \not\equiv +\infty$. By the well-known theorem (Lions and Stampacchia [1967, theorem 2.2]), we obtain the conclusion of our proposition.

Q.E.D.

Proposition 4.3. $\forall q < q_0$, where

$$q_0 = \frac{H^2 - (h_w - h)^2}{2 \, \ell n(r_1/r_0)} \tag{4.10}$$

(4.6) has a unique solution. (4.8) also has a unique solution.

The proof is similar to that of Proposition 4.3. The condition $q < q_0$ ensures that both K_q^* and K_q^{**} are non-empty.

We will prove later that the problems (4.4), (4.6) and (4.8) are equivalent for $q < q_0$ (Theorem 4.13). Now we study (4.4) in detail.

Proposition 4.4. $\forall q \in R$, the solution w_q of (4.4) satisfies, in the sense of distributions, that

$$-1 \leq Lw_q \leq 0 \tag{4.11}$$

$$Lw_q \in L^\infty(D;r) \quad . \tag{4.12}$$

<u>Proof</u>: Given $\psi \in C_0^\infty(D)$, $\psi \geq 0$. Let $v = w_q - \psi$. Then $v \in K_q$

$$v' - w_q' = \begin{cases} -\psi & \text{in} \quad \Omega_1 \cup \{w_q \leq g_{qH}\} \\ w_q - \psi - g_q(r,H) & \text{in} \quad \{g_{qH} < w_q \leq g_{qH} + \psi\} \\ 0 & \text{in} \quad \{w_q > g_{qH} + \psi\} \quad . \end{cases}$$

Hence

$$v' - w_q' \geq \psi \quad . \tag{4.13}$$

On writing (4.5) with $v = w_q - \psi$ and $w = w_q$ we obtain that

$$0 \leq -\int_D r\nabla w_q \cdot \nabla\psi \, drdz - (h_w - h) \int_0^{r_0} r\psi|_{z=h} dr - \int_D r(v' - w_q') drdz$$

$$\leq -\int_D r\nabla w_q \cdot R\psi \, dr\, dz + \int_D r\psi \, drdz = \langle Lw_q + 1, r\psi\rangle$$

for any $\psi \in C_0^\infty(D)$, $\psi > 0$.

Hence
$$Lw_q + 1 \geq 0 \quad . \tag{4.14}$$

Similarly, given $\psi \in C_0^\infty(D)$, $\psi \geq 0$, let $v = w_q + \psi$. Then $v' \geq w_q'$, and (4.5) becomes that

$$0 \leq \int_D r\nabla w_q \cdot \nabla\psi drdz - \int_D r(v' - w_q') drdz \leq \int_D r\nabla w_q \cdot \nabla\psi drdz$$

i.e. $\quad \langle Lw_q, r\psi \rangle \leq 0$ for any $\psi \in C_0^\infty(D)$, $\psi \geq 0$.

Hence
$$Lw_q \leq 0.$$

This inequality and (4.14) prove (4.11), and (4.12)follows from a well-known theorem (Schwartz [1973, th. v, p. 29) and Radon-Nikodyn theorem.

Q.E.D.

Proposition 4.5. If w_q is a solution of (4.4), then

$$\frac{\partial w_q}{\partial n}\Big|_{\Gamma_N} = g_N \quad . \tag{4.15}$$

Proof: At first we prove that, in the sense of distributions,

$$\frac{\partial w_q}{\partial z} = h_w - h \quad \text{on} \quad \Gamma_6 \tag{4.16}$$

$$\frac{\partial w_q}{\partial r} = 0 \quad \text{on} \quad \Gamma_7 \quad . \tag{4.17}$$

Given $\psi \in C_0^\infty(\Gamma_6)$ such that $\psi \geq 0$, and $\varepsilon > 0$, we construct an element $\psi_\varepsilon \in V^1(D)$ with $\psi_\varepsilon \geq 0$ in D, $\psi_\varepsilon = \psi$ on Γ_6, $\psi_\varepsilon = 0$ on Γ_D and

$$\int_D r\psi_\varepsilon \, drdz < \varepsilon \quad . \tag{4.18}$$

Let $v = w_q - \psi_\varepsilon$. Then we have $v' - w_q' \geq \psi_\varepsilon$ similar to (4.13). It follows from (4.5) and generalized Green's formula (Baiocchi and Capelo [1978; Appendix 4 of V.1)

$$0 \leq -\int_D r\nabla w_q \cdot \nabla\psi_\varepsilon drdz + (h_w - h)\int_0^{r_0} r\psi_\varepsilon|_{z=h} dr + \int_D r\psi_\varepsilon \, drdz$$

$$\leq \langle Lw_q, r\psi_\varepsilon\rangle - \langle\frac{\partial w_q}{\partial z}, r\psi_\varepsilon\rangle_{\Gamma_6} + \langle h_w - h, \, r\psi_\varepsilon\rangle_{\Gamma_6} + \langle 1, \, r\psi_\varepsilon\rangle$$

i.e.

$$\langle\frac{\partial w_q}{\partial z} - (h_w - h), \, r\psi_\varepsilon\rangle_{\Gamma_6} \leq \langle Lw_q + 1, \, r\,\psi_\varepsilon\rangle \quad . \tag{4.19}$$

On the other hand, writing (4.4) with $v = w_q + \psi_\varepsilon$ we obtain

$$0 \leq \int_D r\nabla w_q \cdot \nabla\psi_\varepsilon \, drdz - (h_w - h)\int_0^{r_0} r\psi_\varepsilon|_{z=h} dr \qquad (\text{since } v' \geq w_q')$$

$$= -\langle Lw_q, r\psi_\varepsilon\rangle + \langle\frac{\partial w_q}{\partial z}, \, r\psi_\varepsilon\rangle_{\Gamma_6} - \langle h_w - h, \, r\psi_\varepsilon\rangle_{\Gamma_6}$$

i.e.

$$\left\langle \frac{\partial w_q}{\partial z} - (h_w - h),\ r\psi_\varepsilon \right\rangle_{\Gamma_6} \geq \langle L w_q,\ r\psi_\varepsilon \rangle \quad . \tag{4.20}$$

By (4.19), (4.20), (4.11) and (4.18) we have (since $\psi = \psi_\varepsilon$ on Γ_6)

$$\left| \left\langle \frac{\partial w_q}{\partial z} - (h_w - h),\ r\psi \right\rangle_{\Gamma_6} \right| \leq \int_D r\psi_\varepsilon \, drdz < \varepsilon \quad .$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\left\langle \frac{\partial w_q}{\partial z} - (h_w - h),\ r\psi_\varepsilon \right\rangle_{\Gamma_6} = 0 \quad \forall\ \psi \in C_0^\infty(\Gamma_6),\ \psi \geq 0 \quad .$$

This proves (4.16). Now we prove (4.17).

Introduce the notation:

$$F_q = L w_q$$
$$v^* = v(\sqrt{x^2+y^2}, z) \quad \text{for any function} \quad v(r,z)$$
$$D^* = \{(x,y,z) \mid (r,z) \in D,\ r = \sqrt{x^2+y^2}\} \quad \{(x,y,z) \mid x=y=0,\ 0 < z < h\}$$
$$\Gamma_i^* = \{(x,y,z) \mid (r,z) \in \Gamma_i,\ r = \sqrt{x^2+y^2}\}$$
$$\Gamma_D^* = \{(x,y,z) \mid (r,z) \in \Gamma_D,\ r = \sqrt{x^2+y^2}\} \quad .$$

Then D^* is a three-dimensional axisymmetric domain whose boundary is $\Gamma_D^* \cup \Gamma_6^*$, and w_q^* is the solution of the problem:

$$\begin{cases} \Delta w_q^* = F_q^* & \text{in} \quad D^* \\ w_q^* = g_q^* & \text{on} \quad \Gamma_D^* \\ \dfrac{\partial w_q^*}{\partial n} = h_w - h & \text{on} \quad \Gamma_6 \quad . \end{cases}$$

By (4.11), $F_q^* \in L^\infty(D^*)$. Hence $w_q^*|_{D_1} \in H^{2,p}(D_1)$. $p < \infty$, where $D_1 = \{(x,y,z) | (x,y,z) \in D^*, \sqrt{x^2+y^2} < r_0 - \delta\}$. By embedding theorem we have that

$$w_q^* \in C^1(\bar{D}_1) \quad . \tag{4.21}$$

Now it is easily seen that

$$\frac{\partial w_q^*}{\partial x} = \frac{\partial w_q^*}{\partial y} = 0 \quad \text{at} \quad x = y = 0 \quad \text{in} \quad \bar{D}_1 \quad .$$

Hence

$$\frac{\partial w_q}{\partial r} = 0 \quad \text{at} \quad r = 0 \quad \text{in} \quad D$$

(4.17) has been proved. Moreover, (4.21) means that (4.15) is valid in ordinary sense.

Q.E.D.

Now we need the following results (see Chang and Jiang [1978]).

Lemma 4.A. Let $f \in L^p(D;r)$, $p > 2$. Then the problem

$$\begin{cases} Lv = f & \text{in} \quad D \\ v|_{\Gamma_D} = \frac{\partial v}{\partial n}|_{\Gamma_N} = 0 \end{cases}$$

has unique weak solution v in $V^1(D)$, and $v \in C^0(\bar{D})$. Moreover, the linear operator K: $f \mapsto v$, mapping $L^p(D;r)$ $(p > 2)$ into $C^0(\bar{D})$, is compact.

Theorem 4.B. Let

$$U(D) = \{v \in V^2(D) | v = 0 \quad \text{on} \quad \Gamma_D, \frac{\partial v}{\partial n} = 0 \quad \text{on} \quad \Gamma_N\} \quad ,$$

Denote by $R(L)$ the range of the operator L as a map from $U(D)$ into $V^0(D)$. Denote by $R(L)^\perp$ the orthocomplement of $R(L)$ in $V^0(D)$. Then

$$\dim R(L)^\perp = 1$$

i.e. there exists $v_0 \in R(L)^{\perp}$ such that

$$R(L)^{\perp} = \{v \in V^0(D) | v = \mu v_0, \ \mu \in R\} .$$

Remark 4.4. It is easy to show that K is also a compact operator mapping $L^p(D;r)$ into $V^1(D)$.

Now we verify the continuity of the solution of (4.4).

Proposition 4.6. If w_q is a solution of (4.4), then

$$w_q \in C^0(\overline{D}) . \tag{4.22}$$

Proof: Let $f^* = Lw_q - Lv_q$, where v_q is defined by (3.29). Then $f^* \in L^{\infty}(D;r)$, and the problem

$$Lv^* = f^* \qquad \text{in } D$$

$$v^*|_{\Gamma_D} = \frac{\partial v^*}{\partial n}|_{\Gamma_N} = 0$$

has unique solution in $V^1(D)$ which belongs to $C^0(\overline{D})$ (by Lemma 4.A). Clearly, $v^* = w_q - v_q$ is the solution of the problem. Hence $w_q \in C^0(\overline{D})$.

Q.E.D.

Proposition 4.7. Assume w_q is a solution of (4.4). Let

$$\Omega_q = \{(r,z) \in D | r > r_0, \ w_q < g_q(r,H)\} \cup \Omega_1$$

$$\Omega_q^* = \{(r,z) \in D | r > r_0, \ w_q > g_q(r,H)\} .$$

Then, in the sense of distributions,

$$Lw_q = \begin{cases} -1 & \text{in } \Omega_q \\ 0 & \text{in } \Omega_q^* \end{cases} . \tag{4.23}$$

Proof: By (4.22) both Ω_q and Ω_q^* are open sets. Given $\psi \in C_0^{\infty}(\Omega_q)$, define $\psi \equiv 0$ in $D \backslash \Omega_q$. Clearly $E_{\psi} \subset \Omega_q$, where E_{ψ} is the support set of ψ. Let

$$m = \min_{E_\psi \cap \{r \geq r_0\}} (g_q(r,H) - w_q) \quad .$$

Then $m > 0$, and there exists $\lambda^* > 0$ such that for each real λ with $|\lambda| < \lambda^*$ we have

$$|\lambda\psi| < m$$

$$w_q + \lambda\psi < g_q(r,H) \quad \text{in} \quad \Omega_q \cap \{r \geq r_0\} \quad .$$

On choosing in (4.4) $v = w_q + \lambda\psi$, we obtain

$$0 \leq \int_{\Omega_q} r\nabla w_q \cdot \nabla(\lambda\psi)drdz - \lambda \int_{\Omega_q} r\psi\, drdz$$

i.e.

$$\lambda \int_{\Omega_q} r\nabla w_q \cdot \nabla\psi\, drdz \geq \lambda \int_{\Omega_q} r\psi\, drdz \quad .$$

As the sign of λ is arbitrary, we have

$$\int_{\Omega_q} r\nabla w_q \cdot \nabla\psi\, drdz = \int_{\Omega_q} r\psi\, drdz \; \forall\; \psi \in C_0^\infty(\Omega_q)$$

i.e. $Lw_q = -1$ in Ω_q. Similarly, we obtain $Lw_q = 0$ in Ω_q^*.

Q.E.D.

<u>Lemma 4.8</u>. Let $f_2 = Lv_2$, where v_2 is defined by (3.31). If $v \in R(L)^\perp$, then

$$\beta = \int_D vf_2\, r\, drdz \neq 0 \quad . \tag{4.24}$$

<u>Proof</u>: Assume $\beta = 0$. Then $f_2 \perp v$, and $f_2 \perp R(L)^\perp$ (by Theorem 4.B). So $f_2 \in R(L)$. It means that there exists $v^* \in U(D)$ such that

$$\begin{cases} Lv^* = f_2 \\ v^*|_{\Gamma_D} = \frac{\partial v^*}{\partial n}|_{\Gamma_N} = 0 \quad . \end{cases}$$

By (3.38), $f_2 \in L^\infty(D;r)$. Hence $v^* \in C^1(\bar{D})$ (by Theorem 3.A). Let $v = v_2 - v^*$. Then $v \in U(D) \cap C^1(\bar{D})$, and

$$\begin{cases} Lv = 0 & \text{in } D \\ v|_{\Gamma_D} = \begin{cases} 0 & \text{on } \Gamma_1 \cup \Gamma_2 \\ \ln \dfrac{r}{r_1} & \text{on } \Gamma_3 \cup \Gamma_4 \cup \Gamma_5 \end{cases} \\ \dfrac{\partial v}{\partial n}\Big|_{\Gamma_N} = 0 \ . \end{cases}$$

Hence v has minimum in $\overline{D}$, which lies on ∂D; but not on Γ_6 (by Hopf principle); nor on Γ_7 (by remark 3.2). It must be on Γ_D and

$$\min_{\overline{D}} v = v(r_0,h) = \ln \frac{r_0}{r_1} \ .$$

But $\frac{\partial v}{\partial n} = 0$ on Γ_6, then $\frac{\partial u}{\partial n} = 0$ at the point (r_0,h) (as $v \in C^1(\overline{D})$. This contradicts the Hopf principle.

Q.E.D.

<u>Theorem 4.9</u>. Assume v_q is defined by (3.29), $f_q = Lv_q$. w_q is the solution of (4.4), $F_q = Lw_q$. Let $v \in R(L)^{\perp}$,

$$G(q) = \int_D (F_q - f_q)vrdrdz \tag{4.25}$$

then the following two assertions are equivalent:

(1) $\overline{q}$ is a root of the equation

$$G(q) = 0 \tag{4.26}$$

(2) $w_{\overline{q}} \in V^2(D) \cap C^1(\overline{D})$. (4.27)

<u>Proof:</u> If $G(\overline{q}) = 0$, then

$$\int_D (F_{\overline{q}} - f_{\overline{q}})vrdrdz = 0$$

i.e. $(F_{\overline{q}} - f_{\overline{q}}) \perp v$ in $V^0(D)$. By Lemma 4.A we have

$$L(w_{\overline{q}} - v_{\overline{q}}) \in R(L)$$

Hence

$$w_{\overline{q}} - v_{\overline{q}} \in U(D) \ .$$

It follows from Theorem 3.A that

$$w_{\bar{q}} - v_{\bar{q}} \in C^1(\bar{D}) \quad . \tag{4.28}$$

We obtain (r.27). Conversely, if (4.27) is valid, then $F_{\bar{q}} - f_{\bar{q}} \in R(L)$. (4.26 follows from the fact that $v \in R(L)^{\perp}$.

Q.E.D.

<u>Lemma 4.10</u>. (4.26) has at least one real root.

<u>Proof</u>: At first we prove that the function

$$F^*(q) = \int_D F_q \, vrdrdz$$

is continuous in $-\infty < q < +\infty$.

Given $q' \in R$, let $\{q_i\}$ be a sequence converging to q', w_{q_i} be the solution of (4.4) corresponding to q_i, v_{q_i} be defined by (3.29), and $v^*_{q_i} = w_{q_i} - v_{q_i}$. Then we have

$$\begin{cases} Lw_{q_i} = F_{q_i} \in L^\infty(D:r) \\ w_{q_i}|\Gamma_D = g_{q_i} \\ \dfrac{\partial w_{q_i}}{\partial n}|\Gamma_N = g_N \end{cases}$$

$$\begin{cases} Lv^*_{q_i} = F_{q_i} - f_{q_i} \in L^\infty(D;r) \\ v^*_{q_i}|\Gamma_D = 0 \\ \dfrac{\partial v^*_{q_i}}{\partial n}|\Gamma_N = 0 \quad . \end{cases}$$

By (3.29) - (3.32) and (4.11) the sequence $\{F_{q_i} - f_{q_i}\}$ is bounded in $L^2(D;r)$. Therefore it is possible to select a subsequence, still called $\{F_{q_i} - f_{q_i}\}$, in such a way that

$$\{F_{q_i} - f_{q_i}\} \text{ converges weakly to } \overline{F} \text{ in } L^2(D;r) \quad . \tag{4.29}$$

It follows from Lemma 4.A and Remark 4.4 that

$$\{v^*_{q_i}\} \text{ converges strongly to } v^* \text{ in } C^0(\overline{D}) \tag{4.30}$$

$$\{v^*_{q_i}\} \text{ converges strongly to } v^* \text{ in } V^1(D) \tag{4.31}$$

where v^* satisfies

$$Lv^* = \overline{F}$$

$$v^*|_{\Gamma_D} = \frac{\partial v^*}{\partial n}|_{\Gamma_N} = 0 \quad .$$

Since $w_{q_i} = v_{q_i} + v^*_{q_i}$, we have (by (4.30) and (4.31))

$$\{w_{q_i}\} \text{ converges strongly to } w \text{ in } C^0(\overline{D}) \tag{4.32}$$

$$\{w_{q_i}\} \text{ converges strongly to } w \text{ in } V^1(D) \tag{4.33}$$

where $w = v_{q'} + v^*$. Accordingly, $w|_{\Gamma_D} = g_{q'}$ (by (4.32)), i.e. $w \in K_{q'}$. We also have

$$Lw = f_{q'} + \overline{F} \quad . \tag{4.34}$$

Now fix any $v \in K_{q'}$; it is easily seen that there exists a sequence $\{v_i\}$ such that $v_i \in K_{q_i}$ and $\{v_i\}$ converges strongly to v in $V^1(D)$; so that from

$$\int_D r\nabla w_{q_i} \cdot \nabla(v_i - w_{q_i})drdz - (h_w - h)\int_0^{r_0}(v_i - w_{q_i})|_{z=h} rdr$$

$$- \int_D r(v_i' - w_{q_i}')drdz \geq 0$$

it follows (remark that (4.33) implies $w'_{q_i} \to w'$ in $L^1(D;r)$) that

$$\int_D r\nabla w\cdot\nabla(v-w)drdz + (h_w-h)\int_0^{r_0}(v-w)|_{z=h}\, rdr$$
$$- \int_D r(v'-w')drdz \geq 0 \quad .$$

Hence w is the solution of (4.4) for $q = q'$; i.e. $w = w_{q'}$, and (by (4.34))

$$Lw_{q'} = F_{q'} = f_{q'} + \bar{F} \quad . \tag{4.35}$$

By (4.29) and (4.35) we have that

$$\{F_{q_i}\} \text{ converges weakly to } F_{q'} \text{ in } L^2(D;r) \quad .$$

This means that for any $\{q_i\}$ with $\lim_{i\to\infty} q_i = q'$ there exists a subsequence, still called $\{q_i\}$, such that

$$\lim_{q_i\to q'} \int_D F_{q_i}\, vrdrdz = \int_D F_{q'}\, vrdrdz \quad .$$

Accordingly, $F^*(q)$ is continuous. Clearly, $F^*(q)$ is also bounded (by (4.11)).

(4.26) may be rewritten as

$$F^*(q) - \beta q - \alpha = 0 \tag{4.26'}$$

where $\beta = \int_D f_2 vrdrdz \neq 0$ (by Lemma 4.8), $\alpha = \int_D f_1 vrdrdz$. Clearly, the right side of (4.26') changes its sign when q changes from $-\infty$ to $+\infty$. Hence (4.26) has at least one real root.

Q.E.D.

We call the solution of (4.4) regular if $w_q \in C^1(\bar{D}) \cap V^2(D)$. It follows immediately from Lemma 4.10 and Theorem 4.9 that the following theorem is valid.

Theorem 4.11. There exists at least one $\bar{q} \in R$ such that the solution $w_{\bar{q}}$ of (4.4) is regular.

Proposition 4.12. If $w_{\bar{q}}$ is a regular solution of (4.4), then

$$\bar{q} < q_0 \tag{4.36}$$

where q_0 is defined by (4.10).

Proof: Let w_q be the solution of (4.4), and

$$F_1(q,z) = w_q(r_0,z) - w_q(r_0,h) - (h_w-h)(z-h) \quad \text{for} \quad z < h$$

$$F_2(q) = \overline{\lim_{z\to h-0}} \frac{F_1(q,z)}{z-h} .$$

Given $q > q_0$. Assume w_q is regular solution. Then it follows by (4.15) and $w_q \in C^1(\bar{D})$ that

$$F_2(q) = 0 . \tag{4.37}$$

On the other hand, we have

$$\begin{cases} Lw_q \leq 0 & \text{in } D \\ w_q|_{\Gamma_D} = g_q & \\ \frac{\partial w_q}{\partial n}|_{\Gamma_N} = g_N . & \end{cases}$$

Hence w_q has minimum in $\bar{D}$, which lies on ∂D. Clearly, it just is $w_q(r_0,h) \leq 0$. So we have

$$F_1(q,z) \geq -(h_w - h)(z - h)$$

and

$$F_2(q) \leq -(h_w - h) < 0 \quad \text{for} \quad q > q_0 . \tag{4.38}$$

This contradicts (4.37), and w_q cannot be a regular solution for $q > q_0$.

Q.E.D.

To complete this section we display the relation between (4.4), (4.6) and (4.8).

Theorem 4.13. If $q < q_0$, and w_q is the solution of (4.4), then

$$\left.\begin{array}{lll} w_q > 0 & \text{in} & \bar{D} \\ w_q < g_q(r,H) & \text{in} & D \backslash \Omega_1 \end{array}\right\}. \tag{4.39}$$

Moreover, (4.4), (4.6) and (4.8) are equivalent for $q < q_0$.

Proof: By Proposition 4.2 and 4.3 it is sufficient to prove (4.39). Let $q < q_0$ and w_q be the solution of (4.4). It follows from (4.11), (4.15) (3.15) and (3.16) that

$$Lw_q < 0,\ w_q|_{\Gamma_D} > 0,\ \frac{\partial w_q}{\partial n}\Big|_{\Gamma_N} > 0 \ . \tag{4.40}$$

It is easily shown by the maximum principle that $w_q > 0$ in $\bar{D}$. We now prove the second part of (4.39).

Let Ω_q^* be defined as in Proposition 4.7. Noting that

$$w_q = \begin{cases} w_q & \text{in } \Omega_1 \\ \max(w_q, g_q(r,H)) & \text{in } D\backslash\Omega_1 \end{cases}$$

and $w_q \in C^0(\bar{D})$ we may prove that $w'_q \in V^1(D)$ (see for instance Kinderlehrer and Stampacchia [1980, p. 50]). Hence $w''_q \in V^1(D) \cap C^0(\bar{D})$. We have

$$Lw''_q = Lw_q - Lg_q(r,H) = 0 \quad \text{in} \quad \Omega_q^* \quad \text{(by (4.2) and (4.23))}$$

$$w''_q|_{\partial\Omega_q^*} = 0 \quad (\text{by that } w''_q \in C^0(\bar{D}) \text{ and } w''_q|_{\partial D} = 0) \ .$$

It follows from the maximum principle that $w''_q \equiv 0$ in Ω_q^*. Hence $w''_q \equiv 0$ in D and $w_q = w'_q$.

Q.E.D.

It follows from the theorem that

<u>Corollary 4.14.</u> Under the same assumption as in Theorem 4.13 we have

$$\frac{\partial w}{\partial z}\Big|_{\Gamma_D} > 0 \quad . \tag{4.41}$$

5. The Existence of the solution of (PPW).

In this section we prove that a regular solution of (4.4) corresponds to a solution of (PPW). Following the framework of Baiocchi et al. [1973], we establish several lemmas at first.

Throughout this section let w_q be a regular solution of (4.4) and let Ω_q be defined as in Proposition 4.7.

<u>Lemma 5.1.</u>
$$\frac{\partial w_q}{\partial z} \geq 0 \quad \text{in} \quad \bar{D}. \tag{5.1}$$

<u>Proof:</u> Let $E = \{(r,z) \in D \,|\, \frac{\partial w_q}{\partial z} < 0\}$. Then E is an open set, and $E \subset \Omega_q$. In fact, if $(r^*,z^*) \in D \setminus \Omega_q$, then it follows from (4.36) and (4.39) that

$$w_q(r^*,z^*) = g_q(r^*,H) \geq w_q(r^*,z) \quad \text{for} \quad 0 < z < H \quad .$$

Hence $\frac{\partial w_q(r^*,z^*)}{\partial z} = 0$, and $(r^*,z^*) \bar{\in} E$.

If $E \neq \emptyset$, then by Proposition 4.7 we have

$$L\left(\frac{\partial w_q}{\partial z}\right) = 0 \quad \text{in} \quad E \quad .$$

Therefore $\frac{\partial w_q}{\partial z}$ has a strictly negative minimum in $\bar{E}$ (since $w_q \in C^1(\bar{D})$) which lies on ∂E; but neither on $\partial E \cap \Omega_q$ where $\frac{\partial w_q}{\partial z} = 0$; nor on Γ_D (by (4.41)); nor on Γ_6 where $\frac{\partial w_q}{\partial z} = h_w - h > 0$; nor on Γ_7 (by remark 3.2). This is absurd, and $E = \emptyset$.

Q.E.D.

Lemma 5.2. If $q > 0$, then

$$0 < \frac{\partial w_q}{\partial r} < \frac{q}{r} \quad \text{in } D \;. \tag{5.2}$$

Proof. Let $v = r\,\frac{\partial w_q}{\partial r}$. Let $E = \{(r,z)\,|\,v > q\}$. Then E is an open set, and $E \quad \Omega_q$ (since it is easy to show that $v < q$ in $D \backslash \Omega_q$). Therefore,

$$L_1 v = r(Lw_q)_r = 0 \quad \text{in } E$$

where $L_1 = L - \frac{2}{r}\frac{\partial}{\partial r}$ is still an elliptic operator. Simple computation indicates:

$$v = \begin{cases} 0 & \text{on } \Gamma_1 \cup \Gamma_7 \\ q & \text{on } \Gamma_3 \end{cases}$$

$$\frac{\partial v}{\partial n} = 0 \quad \text{on } \Gamma_2 \cup \Gamma_4 \cup \Gamma_5 \cup \Gamma_6 \;.$$

By maximum principle (see for instance Gilbarg and Trundinger [1977]) v has maximum strictly bigger than q in $\bar{E}$ which lies on ∂E. An argument similar to that in the proof of Lemma 5.1 indicates that $E = \emptyset$. Similarly we may prove that $E = \{(r,z)\,|\,v < 0\} = \emptyset$.

Q.E.D.

Remark 5.1. Similarly we have that if $q < 0$ then

$$\frac{q}{r} < \frac{\partial w}{\partial r} < 0 \quad \text{in } D \;. \tag{5.3}$$

Remark 5.2. If $q = 0$, then $\frac{\partial w_0}{\partial r} = 0$. Hence

$$Hz - \frac{z^2}{2} = \frac{H^2}{2} - \frac{(h_w - z)^2}{z} \quad \text{for } h < z < h_w \;.$$

It requires that $H = h_w$. We have assumed that $h_w < H$. So w_0 is not regular solution.

Remark 5.3. By using (5.1), (5.2) and (5.3) we may easily show that if $(r,z) \in D\backslash\Omega_q$ then

$$r \geq r_0, \ z \geq h_w \ .$$

Lemma 5.3.
$$\frac{\partial w_q}{\partial z} = 0 \quad \text{on} \quad \Gamma_3 \ . \tag{5.4}$$

Proof: It is obvious that

$$\frac{\partial w_q(r_1,H)}{\partial z} = \frac{\partial w_q(r_0,H)}{\partial z} = 0 \tag{5.5}$$

Let $q > 0$. Then for $r^* \in [r_0,r_1[$ and $\lambda > 0$ with $r^* + \lambda < r_1$ there exists $\theta \in]0,1[$ such that

$$\left[\frac{\partial w_q(r^*+\lambda,H)}{\partial z} - \frac{\partial w_q(r^*,H)}{\partial z}\right]/\lambda$$

$$= \lim_{h\to+0} \frac{1}{\lambda}\left[\frac{w_q(r^*+\lambda,H-h)-w_q(r^*+\lambda,H)}{-h} - \frac{w_q(r^*,H-h)-w_q(r^*,H)}{-h}\right]$$

$$= -\lim_{h\to+0} \frac{1}{h}\left[\frac{\partial w(r^*+\theta\lambda,H-h)}{\partial r} - \frac{q}{r^*+\theta\lambda}\right] \geq 0 \quad \text{(by Lemma 5.2)} \ .$$

Therefore $\frac{\partial w}{\partial z}$ is nondecreasing function of r on Γ_3, and (5.4) follows from (5.5). The proof for $q < 0$ is similar.

For point $p^* = (r^*,z^*)$ we define the sets

$$Q^+_{p^*} = \{(r,z) \in D \mid r < r^*, \ z > z^*\}$$

$$Q^-_{p^*} = \{(r,z) \in D \mid r > r^*, \ z < z^*\} \ .$$

Lemma 5.4. If $q > 0$, then

$$\overline{Q_p^+} \subset \bar{D}\backslash\bar{\Omega}_q \quad \text{for} \quad p \in \bar{D}\backslash\bar{\Omega}_q \tag{5.6}$$

$$\overline{Q_p^-} \subset \bar{\Omega}_q \quad \text{for} \quad p \in \bar{D} \cap \partial\Omega_q \quad . \tag{5.7}$$

<u>Proof</u>: Let $p^* \in \bar{D}\backslash\bar{\Omega}_q$ and $\alpha(r) = w(r,z^*) - g(r,H)$. We have $r^* > r_0$, $z^* > h_w$ (by Remark 5.3), $\alpha(r^*) = 0$ (Theorem 4.1*), $\alpha'(r) \leq 0$ (by Lemma 5.2). Hence $\alpha(r) \geq 0$ for $r \in [r_0,r^*]$, and $w(r,z) = g(r,H)$ for $r \in [r_0,r^*]$. It follows from Lemma 5.1 and Theorem 4.13 that

$$w(r,z) = g(r,H) \quad \text{in} \quad \{(r,z)|r_0 \leq r \leq r^*, z^* \leq z \leq H\}$$

i.e. $\overline{Q^+_{p^*}} \subset \bar{D}\backslash\Omega_q$.

For $p' = (r',z') \in D\backslash\bar{\Omega}_q$ there exists $p^* \in D\backslash\bar{\Omega}_q$ such that $r^* > r'$, $z^* < z'$. Clearly $\overline{Q^+_{p'}} \subset Q^+_{p^*}$ and $\overline{Q^+_{p^*}} \subset \bar{D}\backslash\Omega_q$. Hence $\overline{Q^+_{p'}} \subset \bar{D}\backslash\bar{\Omega}_q$.

For $p \in \partial D$ (5.6) is trivial. (5.7) is easily seen by reducing to absurdity and (5.6).

Q.E.D.

<u>Remark 5.4</u>. For $q < 0$ and $p^* = (r^*,z^*)$ we define

$$R^+_{p^*} = \{(r,z) \in D|r > r^*, z > z^*\}$$

$$R^-_{p^*} = \{(r,z) \in D|r < r^*, z < z^*\} \quad .$$

Then we obtain by similar argument that

$$\overline{R^+_{p^*}} \subset \bar{D}\backslash\bar{\Omega}_q \quad \text{for} \quad p^* \in \bar{D}\backslash\bar{\Omega}_q$$

$$\overline{R^-_{p^*}} \subset \bar{\Omega}_q \quad \text{for} \quad p^* \in \bar{D} \cap \partial\Omega_q \quad .$$

From Lemma 5.4 immediately follows a property of Ω_q.

<u>Corollary 5.5</u>. Ω_q is a connected set.

<u>Lemma 5.6</u>. $\partial\Omega_q \cap D$ does not contain any vertical or horizontal line segment, and $\partial\Omega_q \cap \Gamma_3 = \emptyset$.

Proof: Assume that $\partial\Omega_q \cap D$ contains a vertical line segment $\Gamma^* = \{(r,z) \in D | r = r', z' < z < z''\}$. Denote $N_1 = \{(r,z) \in D | r > r', z' < z < z''\}$, $N_2 = \{(r,z) \in D | r < r', z' < z < z''\}$. Then $N_1 \subset \bar{\Omega}_q$ and $N_2 \subset D \backslash \bar{\Omega}_q$ (by Lemma 5.4). Hence $w_q(r,z) = g_q(r,H)$ and $\frac{\partial w_q}{\partial r} = q/r$ in $\bar{N}_2$; $Lw_q = -1$ in N_1. Therefore $w_q|_{\bar{N}_1}$ is the solution of the Cauchy problem

$$\begin{cases} Lw_q = -1 & \text{in } N_1 \\ w_q|_{\Gamma^*} = g(r',H) & \\ \frac{\partial w_q}{\partial r}|_{\Gamma^*} = \frac{q}{r'} . & \end{cases}$$

By the uniqueness of the solution we have

$$w_q = \frac{r'^2}{2} \ln \frac{r}{r'} + \frac{1}{4}(r'^2 - r^2) + g_q(r,H) \quad \text{in } \bar{N}_1 .$$

But $w_q = Hz - \frac{z^2}{2}$ on Γ_2. This contradiction proves that $\partial\Omega_i \cap D$ does not contain any vertical line segment. By similar argument and (5.4) we obtain that $\partial\Omega_q \cap D$ does not contain any horizontal line segment and $\partial\Omega_q \cap \Gamma_3 = \emptyset$.

Q.E.D.

Theorem 5.7. If $q < 0$, and

$$\Omega_q = \Omega_1 \cup \{(r,z) \in D | r > r_0, w_q < g_q(r,H)\} \tag{5.8}$$

$$\varphi_q(r) = \sup\{z | (r,z) \in \Omega_q\} \quad \text{for } r \in]r_0, r_1[\tag{5.9}$$

$$\varphi_q(r_0) = \lim_{r \to r_0+0} \varphi_q(r), \; \varphi_q(r_1) = \lim_{r \to r_1-0} \varphi_q(r) \tag{5.10}$$

$$\bar{u}_q = \frac{\partial w_q}{\partial z} + z \quad \text{in} \quad \bar{D}, \; u_q = \bar{u}_q|_{\Omega_q} \tag{5.11}$$

then $\{u_q, \varphi_q(r)\}$ is the solution of (PPW).

<u>Proof</u>: First we note that $\varphi_q(r)$ is a well-defined, strictly increasing, continuous function for $r \in]r_0,r_1[$. In fact, for any $r \in]r_0,r_1[$ we have $w_q < g_q(r,H)$ if z is small enough since $w_q = 0$ on Γ_1 and $g_q(r,H) > 0$. So $\{z|(r,z) \in \Omega_q\}$ is nonempty and $\varphi_q(r)$ is well-defined. It immediately follows from (5.1) and the definition of $\varphi_q(r)$ that

$$\Omega_q = \Omega_1 \cup \{(r,z) \in D|r > r_0, \; 0 < z < \varphi_q(r)\} \quad . \tag{5.12}$$

Lemma 5.6 shows that $\{(r,\varphi_q(r))|r_0 < r < r_1\}$ is a Lipschitz graph with respect to the axes $\bar{x} = r-z$, $\bar{y} = r+z$. Hence $\varphi_q(r)$ is a strictly increasing, continuous function.

By virtue of (5.9), (5.10) and (3.15) it is readily shown that $\varphi(r_0) > h_w$, $\varphi(r_1) = H$. Then (2.6) and (2.7) have been proved. (2.8) is obvious.

Now we check (2.4). Since $w_q = g_q(\cdot,H)$ in $\bar{D}/\bar{\Omega}_q$ we have

$$\frac{\partial w_q}{\partial z} = 0, \; u_q = z \quad \text{on} \quad \Gamma_0$$

the rest of (2.4) is obvious thanks to (3.15) and (3.16).

Finally we check (2.9). Given any $\psi \in C^2(\bar{\Omega}_q)$ with $\psi = 0$ in a neighborhood of $\Gamma_2 \cup (\Gamma_4 \cap \partial\Omega_q) \cup \Gamma_5 \cup \Gamma_6$ we have (note (4.23) and (5.12))

$$\int_{\Omega_q} r\nabla u_q \cdot \nabla\psi drdz = \int_{\Omega_q} r[\frac{\partial^2 w_q}{\partial r \partial z}\frac{\partial\psi}{\partial r} + (\frac{\partial^2 w_q}{\partial z^2} + 1)\frac{\partial\psi}{\partial z}]drdz$$

$$= \int_{\Omega_q} [\frac{\partial}{\partial z}(r\frac{\partial w_q}{\partial r})\frac{\partial\psi}{\partial r} - \frac{\partial}{\partial r}(r\frac{\partial w_q}{\partial r})\frac{\partial\psi}{\partial r}]drdz$$

$$= \int_{\Omega_q} [\frac{\partial}{\partial z} (r \frac{\partial w_q}{\partial r} \frac{\partial \psi}{\partial r}) - \frac{\partial}{\partial r} (r \frac{\partial w_q}{\partial r} \frac{\partial w}{\partial z})] drdz$$

$$= - \int_{\partial\Omega_q} r \frac{\partial w_q}{\partial r} \frac{\partial \psi}{\partial r} dr + r \frac{\partial w_q}{\partial r} \frac{\partial \psi}{\partial z} dz$$

$$= - \int_{\Gamma_0} r \frac{\partial w_q}{\partial r} \frac{\partial \psi}{\partial r} dr + r \frac{\partial w_q}{\partial r} \frac{\partial \psi}{\partial r} dz \quad (\text{since } \frac{\partial w_q}{\partial r} = 0 \text{ on } \Gamma_1 \cup \Gamma_7)$$

$$= -q \int_{\Gamma_0} \frac{\partial \psi}{\partial r} dr + \frac{\partial \psi}{\partial z} dz \quad (\text{since } w_q = g_q(r,H),\ r\frac{\partial w_q}{\partial r} = q \text{ in } D\backslash\Omega)$$

$$= -q[\psi(r_0,\varphi(r_0)) - \psi(r_1,\varphi(r_1)) + \psi(r_0,\varphi(r_0)) - \psi(r_1,\varphi(r_1))] = 0 \quad .$$

The proof is completed by virtue of the denseness of $\{\psi\}$ described above in K_1.

Q.E.D.

If $q < 0$, then by using similar arguments we obtain that $\varphi_q(r)$ is strictly decreasing, continuous for $r \in]r_0,r_1[$, and that $\varphi_q(r_1) = H$. This is absurd. Hence we obtain (recall (4.36) and remark 5.2):

Proposition 5.8. If w_q is a regular solution, then

$$0 < q < q_0 \quad . \tag{5.13}$$

Proposition 5.9. Let

$$Q^* = \{q | w_q \text{ is a regular solution of (4.4)}\} \quad . \tag{5.14}$$

Then:

$$Q^* \subset]0,q_0[\tag{5.15}$$

$$Q^* \text{ is a closed set} \tag{5.16}$$

$$w_q \text{ is nonincreasing on } Q^* \tag{5.17}$$

Proof: (5.15) is clear by virtue of (5.13). (5.16) follows immediately from Theorem 4.9 and the continuity of $G(q)$ (see the proof of Lemma 4.10). Now we prove (5.17). Let q_1, $q_2 \in Q^*$, $q_1 < q_2$, and

$$E = \{(r,z) \in D \mid w = w_{q_1} - w_{q_2} < 0\} \quad .$$

Then $w_{q_1} < w_{q_2} < g_{q_2}(r,H) < g_{q_1}(r,H)$ in $E \cap \{r > r_0\}$. Hence $E \subset \Omega_{q_1}$, and $Lw < 0$ in E (by Proposition 4.4 and 4.7). w has strictly negative minimum on $\bar{E}$ which lies on ∂E; but not on $\partial E \cap D$ where $w = 0$; nor on Γ_D where $w > 0$; nor on Γ_6 where $\frac{\partial w}{\partial n} = 0$; nor on Γ_7 (by remark 3.2). This is absurd. Hence $E = \emptyset$.

Q.E.D.

Let $q_m = \inf_Q\{q\}$, $q_M = \sup_Q\{q\}$. By (5.15) and (5.16) we have

$$q_m, q_M \in Q^*, \; q_m > 0, \; q_M < q_0 \quad .$$

From (5.17) follows immediately the theorem

Theorem 5.10. For any $q \in Q^*$ we have

$$w_{q_m} > w_q > w_{q_M} \quad \text{in } \bar{D} \quad . \tag{5.18}$$

APPENDIX

The proof of Theorem 3.A

Taken δ with $0 < \delta < r_0$. Let

$$\omega_1 = \{(r,z) \in D \mid 0 < r < \delta\}$$

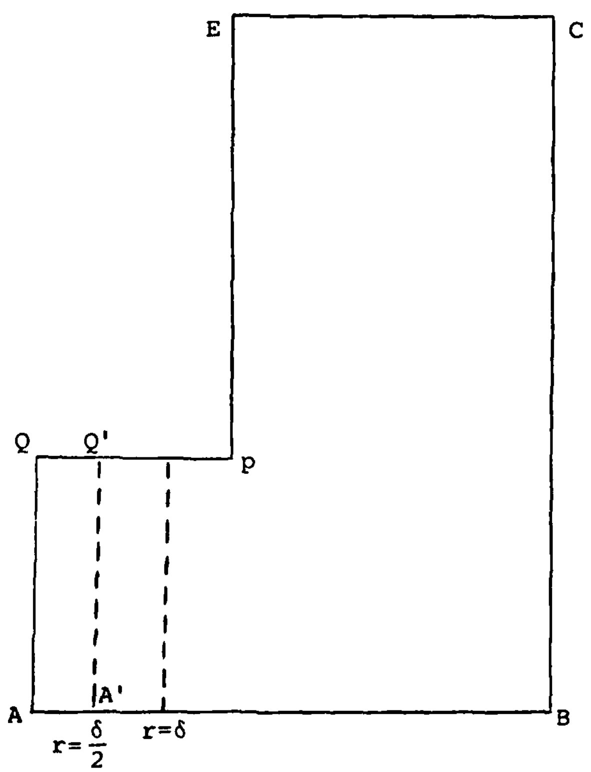
$$\omega_2 = \{(r,z) \in D \mid r > \delta/2\}$$

then $D = \omega_1 \cup \omega_2$. Assume ρ_1, ρ_2 is the corresponding partition of unity, then

$$\rho_1 + \rho_2 \equiv 1 \quad \text{for} \quad (r,z) \in D \ .$$

We may choose ρ_2 such that

$$\rho_2 = \begin{cases} 1 & r > \delta \\ 0 & r < \dfrac{\delta}{2} \end{cases} \ . \tag{1}$$



It is to show that

$$
\begin{cases} L(\rho_1 v) = f\rho_1 + vL\rho_1 + 2\nabla\rho_1\cdot\nabla v \\ \rho_1 v\big|_{\Gamma_D\cap\bar{\omega}_1} = \dfrac{\partial}{\partial n}(\rho_1 v)\Big|_{\Gamma_N\cap\bar{\omega}} = \rho_1 v\big|_{r=\delta} = 0 \ . \end{cases} \tag{2}
$$

It follows from $v \in V^2(D)$ that $\nabla v \in V^1(\omega_1)$. Hence

$$\nabla v \in W_2^1(\tilde{\omega}_1)$$

where $\tilde{\omega}_1$ is three-dimensional domain obtained by rotating ω_1 around z-axis. By the embedding theorem we have

$$\nabla v \in L^p(\tilde{\omega}_1) \qquad (p < 6) \ .$$

So $\nabla v \in L^p(\omega_1; r)$ and

$$L(\rho_1 v) \in L^p(\omega_1; r) \qquad (p < 6) \ .$$

Considering $\rho_1 v$ as the solution of three-dimensional problem (2), we have

$$\rho_1 v \in W_p^2(\tilde{\omega}_1) \qquad (p < 6) \ .$$

By using the embedding theorem again and returning to two-dimensional dmain we obtain

$$\rho_1 v \in C^\beta(\bar{\omega}_1) \qquad (\beta < \tfrac{3}{2}) \ . \tag{3}$$

In ω_2 = polygon A'BCEPQ' the operator L is non-singular. It is easily seen that

$$
\begin{cases} \Delta(\rho_2 v) = g \\ \rho_2 v\big|_{\Gamma_D\cap\bar{\omega}_2} = \dfrac{\partial(\rho_2 v)}{\partial n}\Big|_{\Gamma_N\cap\bar{\omega}_2} = \rho_2 v\big|_{r=\frac{\delta}{2}} = 0 \end{cases}
$$

where

$$g = \rho_2 f + vL\rho_2 + 2\nabla\rho_2\cdot\nabla v - \frac{1}{r}\frac{\partial}{\partial r}(\rho_2 v) \in L^{p_1}(\omega_2) \qquad (\rho_1 < 6) \ .$$

Let $v_0 = g * \frac{1}{2\pi}\ln\frac{1}{\sqrt{r^2+z^2}}$, where $*$ expresses convolution operation, then $\Delta v_0 = g$ and

$$v_0 \in W^2_{p_1}(\omega_2) \subset C^k(\bar{\omega}_2) \qquad (k < \tfrac{5}{3}) \ .$$

Let $v_1 = v_0 - \rho_2 v$, then

$$\begin{cases} \Delta v_1 = 0 \\ v_1|_{\partial w_2 \setminus pQ'} = v_0|_{\partial w_2 \setminus pQ'} \\ \frac{\partial v_1}{\partial n}|_{pQ'} = \frac{\partial v_0}{\partial n}|_{pQ'} \ . \end{cases}$$

By assumption $v \in V^2(D)$ we have

$$\rho_2 v \in W^2_2(\omega_2) \subset C^{k'}(\omega_2) \qquad (0 < k' < \tfrac{1}{2}) \ .$$

Hence

$$v_1 \in C^{k'}(\omega_2) \qquad (0 < k' < \tfrac{1}{2}) \ .$$

By using theorem 4.4 of Volkov [1965] (Trudy of Mathematics Institut of Steklov, 77 (1965), 113-142) we have

$$v_1 \in C^{\beta}(\bar{\omega}_2) \qquad (\beta < \tfrac{5}{3}) \ .$$

So

$$\rho_2 v \in C^{\beta}(\bar{\omega}_2) \qquad (\beta < \tfrac{5}{3}) \ . \tag{4}$$

It follows from (3) and (4) that

$$v \in C^{\beta}(\bar{D}) \qquad (\beta < \tfrac{3}{2}) \ .$$

Q.E.D.

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER #2245	2. GOVT ACCESSION NO. AD-A103	3. RECIPIENT'S CATALOG NUMBER 858
4. TITLE (and Subtitle) The Solution of the Free Boundary Problem for an Axisymmetric Partially Penetrating Well		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) C. W. Cryer and S. Z. Zhou		8. CONTRACT OR GRANT NUMBER(s) MCS77-26732 DAAG29-80-C-0041
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin, 610 Walnut Street, Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 1 - Applied Analysis
11. CONTROLLING OFFICE NAME AND ADDRESS See Item 18 below		12. REPORT DATE July 1981
		13. NUMBER OF PAGES 47
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

and

National Science Foundation
Washington, D. C. 20550

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Free Boundary problem; axisymmetric well; weighted Sobolev spaces; families of variational inequalities; existence.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The weak form of the free boundary problem for an axisymmetric partially penetrating well may be formulated as follows: find $\varphi(r) \in C^0([r_0,r_1])$ and $u \in C^0(\bar{\Omega}) \cap V^1(\Omega)$ such that

$$\int_\Omega r\nabla u\cdot\nabla v\,drdz = 0 \quad \text{for all} \quad v \in K_1$$

and u satisfies appropriate boundary conditions. Here, u is related to the hydraulic head, $\varphi(r)$ is the unknown water-air interface, Ω is the region of saturated flow

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ABSTRACT (continued)

$$\Omega = \{(r,z) | 0 < r \leq r_0, \ 0 < z < h\} \cup \{(r,z) | r_0 < r < r_1, \ 0 < z < \varphi(r)\} \ ,$$

K_1 is a convex set in the weighted Sobolev space $V^1(\Omega)$.

We reduce the problem to three families of variational inequalities by using a type of "Baiocchi transform", study equivalence of the three families and regularity of the solutions of the variational inequalities. Finally, we prove the existence of the solution for the well problem.